Mat 320 - Stout Name:______________________

QUIZ 14: ABSTRACT ALGEBRA

Problem 1. Let k be a field and let $R = k[x]$ be the ring of univariate polynomials over k.

Part A: Circle all that apply:

- R is an integral domain
- R is integrally closed
- R is an unique factorization domain (UFD)
- R is a principal ideal domain (PID)
- R is an euclidean domain (ED)
- R is a field

Part B: We say that a field k is *algebraically closed* if every non-constant polynomial $p(x)$ in $k[x]$ has a root in k. Having a root in K means that there is an $a \in k$ such that $p(a) = 0$. Why is $\mathbb{C}$ algebraically closed but $\mathbb{R}$ is not?

Part C: Show that the ideal $(x^2 + 1)$ a maximal ideal of $\mathbb{R}[x]$. Can you describe other maximal ideals?

Problem 2. Since $I = (x^2 + 1)$ is a max ideal of $\mathbb{R}[x]$, the ring $\mathbb{R}[x]/I$ is a field.

Part A: Construct a surjective morphism $\mathbb{R}[x] \to \mathbb{C}$ whose kernel is I and use the first isomorphism theorem to prove $\mathbb{R}[x]/I \cong \mathbb{C}$.

Part B: Describe the field $\mathbb{Q}(\sqrt{2})$ as factor ring of $\mathbb{Q}[x]$?

Part C: Let R be a PID. An ideal $I = (p)$ is prime[1] if and only if p is irreducible (i.e., doesn't factor) in R. What are the prime ideals of $\mathbb{C}[x]$? What are the prime ideal of $\mathbb{R}[x]$?

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[1]A prime ideal is any proper ideal I of R such if for all $a, b \in R$, $a \cdot b \in I \implies a \in I$ or $b \in I$. If R is a commutative ring, the collection of all prime ideal is called the *Spectrum of the ring* R and it is denoted by $\mathrm{Spec}(R)$ and the subset of $\mathrm{Spec}(R)$ given by all maximal ideals is denoted by $\mathrm{MSpec}(R)$. Alternatively, this subset may be described as the collection of all closed points of the non-T_1 topological space $\mathrm{Spec}(R)$. The space $\mathrm{Spec}(R)$ and its generalizations are what algebraic geometers study!