Mat 320 - Stout Name:____________________

QUIZ 13: ABSTRACT ALGEBRA

Problem 1: Let $(R, +, \dot{})$ be a ring – i.e., satisfies our 8 properties plus the two distributive properties. We say that R is an *integral domain* if it has no non-trivial zero divisors–i.e., whenever $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Part A. Let $R = M_2(\mathbb{R})$ be the collection of all 2×2 matrices with real entries. Explicitly show that R is not an integral domain.

Part B. Assume further that R is a commutative ring. Construct an example of a non-domain by finding a non-prime ideal $I \trianglelefteq R$ and then modding out to form the ring R/I.

Problem 2: Let R_1 and R_2 be two rings and define the *direct sum* of R_1 and R_2 to be the ring given by

$$R_1 \oplus R_2 := \{(r_1, r_2) \mid r_i \in R_i, i = 1, 2\}$$

with addition defined by $(r_1, r_2) + (r_1', r_2') = (r_1 + r_1', r_2 + r_2')$ and multiplication defined by $(r_1, r_2) \cdot (r_1', r_2') = (r_1 \cdot r_1', r_2 \cdot r_2')$.

Part A: Show that $R_1 \oplus R_2$ is a ring[1]. What is the multiplicative identity in this ring?

Part B: Construct ring homomorphisms $R_1 \oplus R_2 \to R_i$.

Part C: Why is the group homomorphism $R_1 \to R_1 \oplus R_2$ given by $x \mapsto (x, 0)$ not a ring homomorphism? Is there an injective ring homomorphism $R_1 \hookrightarrow R_1 \oplus R_2$ for non-trivial rings R_i?

Problem 3 We say that a ring R is a *principal ideal domain* (PID) if it is an integral domain such that every ideal is generated by a single element – i.e., if I is an ideal of R, then there exists a $p \in R$ such that $I = (p)$.

Part A: Show that $\mathbb{Z}$ is a PID.

Part B: Show that $\mathbb{Z}[x]$ is not a PID.

---

[1] You do not need to demonstrate the first 5 properties of a ring because we already established them in the group theory section of the course