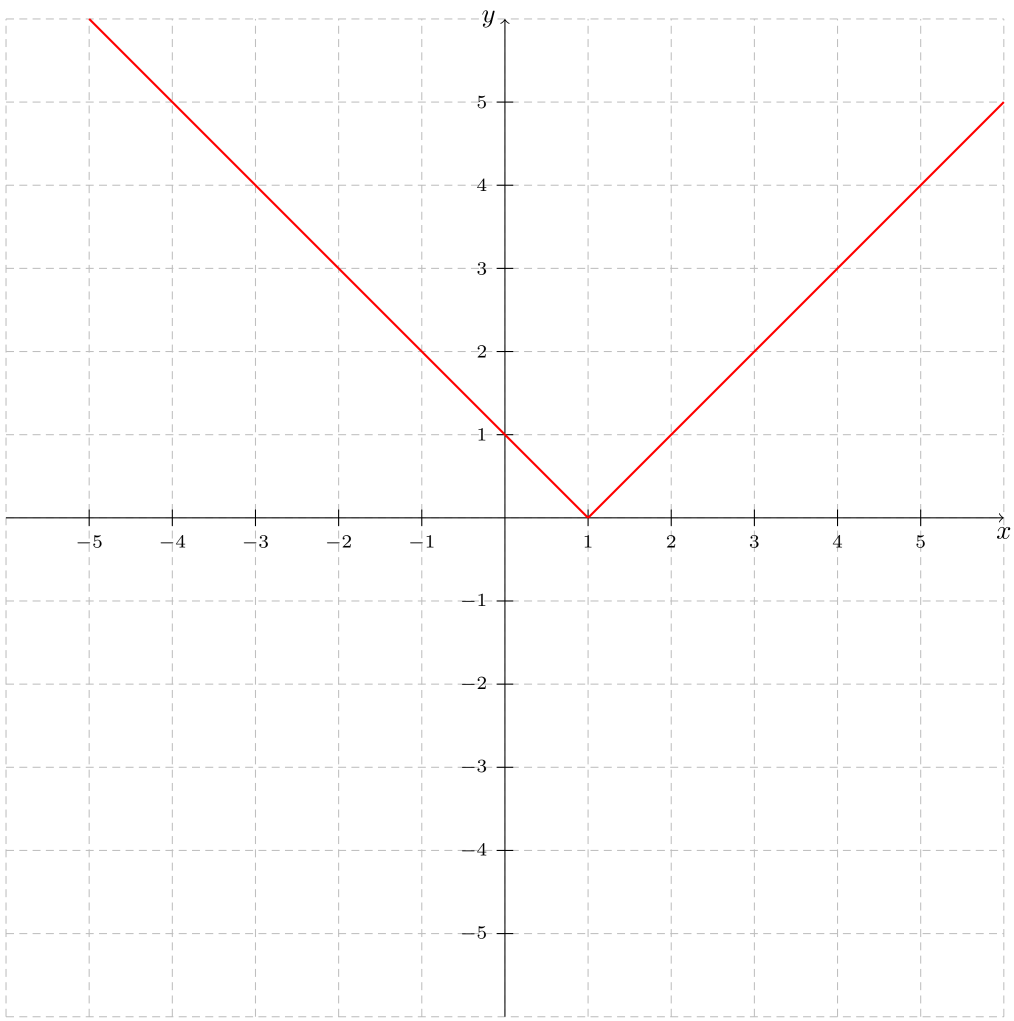
PRACTICE WITH RIEMANN SUMS

Let $f(x)$ be a continuous function on an closed interval $[a, b]$. We are going to practice approximating the definite integral $\int_a^b f(x)dx$ via Riemann Sums by hand.

1. Example: Absolute Value

Lets first start with the function $f(x) = |x - 1|$ and we will use limits of integration $a = 0$ and $b = 5$.

STEP 1: Using the graph below and number of partitions $N = 5$, draw each rectangle needed to form the (left) Riemann sum: *Graph of* $y = |x - 1|$



STEP 2: Calculate the area of each rectangle using the formula $area = length \times height$ and write the result in each space below:

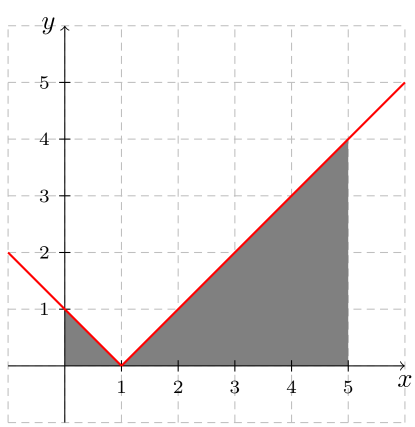
Area of Rect. 1	Area of Rect. 2	Area of Rect. 3	Area of Rect. 4	Area of Rect. 5

STEP 3: Add up the 5 areas above and place it to the right of the approximation symbol below

$$\int_0^5 |x-1|dx \approx \underline{\qquad\qquad} \tag{1}$$

REMARK 1.1. We can actually find an exact answer for this integral without even using the concepts of Riemann Sums or Definite Integrals. This is because the function in question is not curved and the area we are interested in is made of two triangles as displayed below:

Zoomed in Graph of $y = |x-1|$



So, this area can be actually calculated using the formula for the area of a triangle $area = \frac{1}{2} base \times height$ to calculate the area of the two triangles:

Area of Tri. 1	Area of Tri. 2

Thus, the exact value of the integral is

$$\int_0^1 |x-1|dx = \underline{\qquad\qquad}, \tag{2}$$

which we can compare with our approximation above.

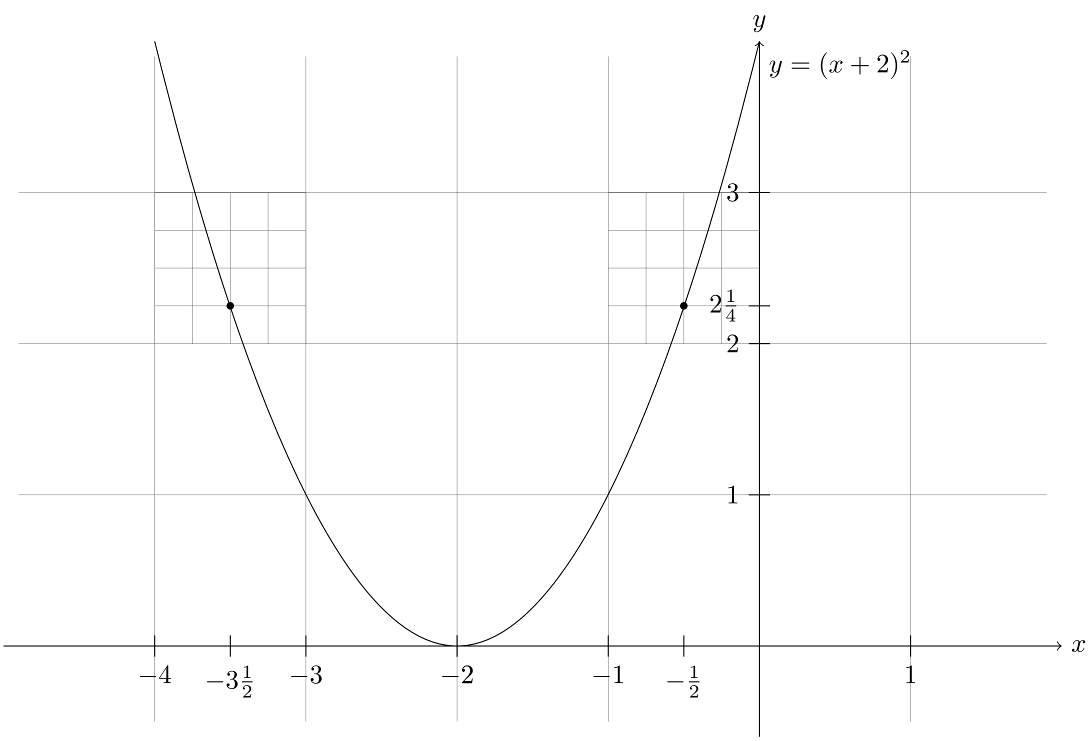
This remark highlights a potential yet tentative step 4. Is there a way to simplify our Riemann sums or another method for calculating the integrals directly so that we can have an exact value for our integral instead of the approximate value given by our Riemann sums.

2. Example: Quadratic Function

We will perform the same sequence of steps but this time we will use the function $f(x) = (x+2)^2 = x^2 + 4x + 4$ defined on the closed interval $[-4, 0]$.

This time we only use 4 partitions (instead of 5 like we did last time) and we will also use the midpoint method this time.

Step 1: Draw the four rectangles starting from -4 to 0 each of width 1 and whose height is determined by the value of the midpoint – i.e., the heights will be $f(-3.5)$, $f(-2.5)$, $f(-1.5)$, $f(-0.5)$.



STEP 2: Calculate the area of each rectangle using the formula $area = length \times height$ and write the result in each space below:

Area of Rect. 1	Area of Rect. 2	Area of Rect. 3	Area of Rect. 4

STEP 3: Add up the 4 areas above and place it to the right of the approximation symbol below

$$\int_{-4}^{0} (x+2)^2 dx \approx \underline{\qquad\qquad} \tag{3}$$

Our tentative step 4: Every quadratic function has an axis of symmetry at its vertex $x = c$. If we integrate such a function $f(x)$, or any function with axis of symmetry at $x = c$, then we have the following simplification of the definite integral with limits $a = c - t$ and $b = c - t$:

$$\int_{c-t}^{c+t} f(x)dx = \int_{c-t}^{c} f(x)dx + \int_{c}^{c+t} f(x)dx = 2\int_{c}^{c+t} f(x)dx \tag{4}$$

And, if $F(x)$ is the antiderivatve of $f(x)$, then we have the exact equation

$$\int_{c-t}^{c+t} f(x)dx = 2[F(c+t) - F(c)] \tag{5}$$

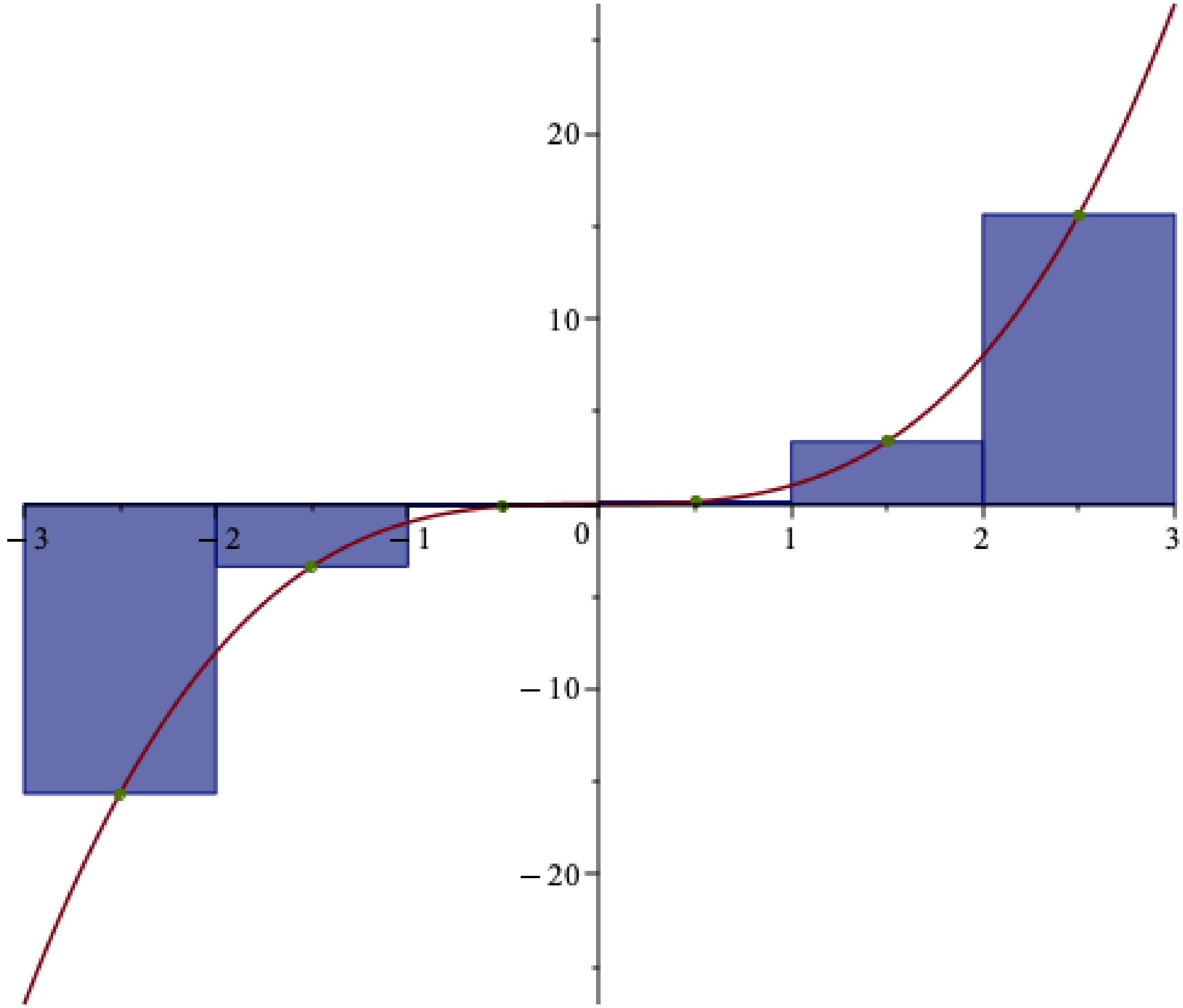
We will get to how indefinite integrals (i.e., antiderivatives) relate to definite integrals and therefore Riemann sums later. But, as far as Equation 4, we can actually notice this at the level of Riemann sums – i.e., the sums simplify to two terms:

$$\underset{\text{Area of Rect. 1 \& 4 together}}{\underline{2 \times \boxed{}}} \qquad \underset{\text{Area of Rect. 2 \& 3 together}}{\underline{2 \times \boxed{}}}$$

3. Example: Cubic Function

In this example, we are going to carry out the previous three steps but now with the function $f(x) = x^3$ on the closed interval $[-3, 3]$. This time we do 6 partitions and we keep with the midpoint method. This time lets ask maple for help:

Step 1:



A midpoint Riemann sum approximation of $\int_{3}^{3} f(x)\, dx$, where $f(x) = x^3$ and the partition is uniform. The approximate value of the integral is 0.. Number of subintervals used:6.

According to maple this Riemann sum adds up to zero, but lets check by carrying out the next step.

Step 2:

Area of Rect. 1	Area of Rect. 2	Area of Rect. 3	Area of Rect. 4	Area of Rect. 5	Area of Rect. 6

As we can see we have three "negative areas" and three "positive areas", so it is true:

Step 3:

$$\int_{-3}^{3} x^3 dx \approx \underline{\qquad\qquad} \tag{6}$$

Step 4: Explanation of what is happening in this example:

Well now we have to try Step 4 and see what is going on here. Just like in the previous example, we have a point of symmetry on the graph which is effecting the results. We could generalize this to any "axis of antisymmetry", but lets put this in terms of integrating an odd function.

Recall an odd function is functions $f(x)$ such that $f(-x) = -f(x)$ and this crucial facts helps us integrate any odd function on the interval $(-t, t)$ for any positive number t

(7)
$$\begin{aligned}
\int_{-t}^{t} f(x)dx &= \int_{-t}^{0} f(x)dx + \int_{0}^{t} f(x)dx \\
&= \int_{-(-t)}^{-0} f(-x)(-1)\cdot dx + \int_{0}^{t} f(x)dx \quad \text{A result of substitution } -x \text{ for } x \\
&= -\int_{t}^{0} f(-x)dx + \int_{0}^{t} f(x)dx \\
&= -(-\int_{0}^{t} f(-x)dx) + \int_{0}^{t} f(x)dx \quad \text{A general property of integrals} \\
&= \int_{0}^{t} -f(x)dx + \int_{0}^{t} f(x)dx \quad \text{Because } f(x) \text{ is odd} \\
&= -\int_{0}^{t} f(x)dx + \int_{0}^{t} f(x)dx \\
&= 0
\end{aligned}$$

Applying this to our function verifies that the Riemann sum above is not just an approximation, but it is actually exact. Moreover, we can also verify it using indefinite integral since $\frac{x^4}{4} + c$ is the antiderivative of x^3, we can also calculate

$$\int_{-3}^{3} x^3 dx = \frac{x^4}{4}\Big|_{-3}^{3} = \frac{3^4}{4} - \frac{(-3)^4}{4} = \frac{81}{4} - \frac{81}{4} = 0 \tag{8}$$

Finally, and maybe more elegantly, it doesn't matter how many partitions we choose using the midpoint method as long as there is an even number of them. Thus, for $N = 2k$, the associated Riemann sum will sum to zero. Since we know the sum converges to a number, this number must also be zero.

4. Example 4: A limit of Riemann Sums of x^2

In the example, we will look at the quadratic function $f(x) = x^2$ as defined on $[0, 1]$ and we will perform a limit of Riemann Sums R_n. If such a limit exists, we call it the Riemann Integral of $f(x)$. The general point is that we have agreement amoung three:

$$\lim_{n\to\infty} R_n = \int_0^1 f(x)dx = [\int f(x)dx]_0^1 \tag{9}$$

The right side is the easiest to calculate:

$$[\int x^2 dx]_0^1 = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}. \tag{10}$$

Almost all the time, you will calculate integrals as done above. But, this one time, we need to understand the Riemann Integral definition of integral and actually compute it.

<u>Step 0</u>: Here we need to add a step because we have to be a lot more carefully choosing our partition and its associated intervals. We want a uniform partition of $[0, 1]$ each of length $\frac{1}{N}$ where N is still the number of partitions. This looks like

$$0 < \frac{1}{N} < \frac{2}{N} < \frac{3}{N} < \cdots < \frac{N-1}{N} < \frac{N}{N} = 1. \tag{11}$$

So, the ith general interval (for $i = 1, 2, 3, \ldots, N$) is $[\frac{i-1}{N}, \frac{i}{N}]$.

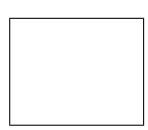
Furthermore, we need to pick the easiest points x_i^* in each interval to work with. In this case, it is best to work with a right Riemann sum, which means we will choose $x_i^* = \frac{i}{N}$ for $i = 1, 2, 3, 4, \ldots, N$.

<u>Step 1</u>: We should draw a graph, but this time we have N rectangles. So it is hard to draw it perfectly. Here we will just sketch a graph.

<u>Step 2</u>: We write down the area of each rectangle. Since the partition is uniform, we know that the width of each rectangle is constant:

$$width = \frac{1}{N}$$

and so we just need to figure out the height:

$f(x_i^*) \cdot \frac{1}{N} =$ 

Area of Rect. i

Step 3: Now, we need to add up all the areas (there are N of them) in which case we have

$$R_N := \sum_{i=1}^{N} f(x_i^*) \cdot \tfrac{1}{N} = \underbrace{\sum_{i=1}^{N} \boxed{\phantom{\rule{2cm}{1.5cm}}}}_{\text{Sum of Areas of all Rectangles}}$$

Step 3.5: The main problem with using Riemann Sums to calculate an integral as opposed to them approximating is that finding a formula for the sums is usually not easy. In this case, we need to use a special case of a deep result in mathematics known as **Faulhaber's formula**. In our case, it gives us the formula:

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \tag{12}$$

Applying this in our case gives

$$R_N = \frac{1}{N^3} \sum_{i=1}^{N} i^2 = \frac{1}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2} \tag{13}$$

Step 3.75: Now, in fact, in this form it is finally easy to take the limit:

$$\int_0^1 x^3 dx = \lim_{n \to \infty} R_N = \lim_{n \to \infty} \left[\frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}\right] = \frac{1}{3} \tag{14}$$

Step 4: There is only one way to simplify this computation and that is to learn as many antiderivatives as we can and calculate integrals as we did at the beginning of this example.

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